**Chapter 3: Situations**

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Two of the defining characteristics of the Mathematical Understanding for Secondary Teaching (MUST) Framework are its grounding in teaching practice and its focus on mathematics teaching at the secondary level. Our strategy for remaining faithful to practice at the secondary level was to observe and record interesting mathematical events that took place in mathematics classes taught in grades 6–12. These events often involved interesting student questions, teacher-created tasks, or mathematical conjectures and discussions. We recognized that interesting, relevant discussions took place in school hallways or teachers’ workrooms as well as in university classes for students preparing to teach mathematics at the secondary level. Records of these events were turned into formal documents called Situations, which became the data that informed the construction of the MUST Framework. Because it is based on data from events that actually happened, the MUST Framework captures and organizes mathematical understanding that is useful for secondary teachers in their practice

Structure of a Situation

A Situation is a formal document that contains three main parts: a prompt, a set of foci, and commentary. Each of the components is discussed in the following sections. Examples of complete Situations can be found in chapters 7–7+*n*.

Prompt

**Prompt**. The prompt sets the stage for the mathematics of the Situation by briefly describing an event from teaching practice. Teaching practice includes preparing, implementing, and reflecting on classroom instruction. The instruction can be either in secondary mathematics or in the preparation and professional development of secondary teachers.

*Figure 1*: A working description of a prompt

The prompts are the heart of the Situation and the direct ties of the Situations to practice. A prompt, as described in Figure 1, should stimulate interesting mathematical discussion and may arise from student and teacher insights as well as from difficulties and questions. A prompt can be viewed from many levels and it often takes multiple readings to appreciate the complexity of the mathematics that would be useful to understand if one is to make the most of the opportunity to bring varied mathematical ideas to bear on events like that captured by the prompt.

The prompt in Figure 2 is authentic and most algebra teachers can relate to this event, which is the basis of the Solving Quadratic Equations Situation (#35). It is also good because it can be unpacked to expose important mathematics that generalizes to many events in mathematics classrooms at the secondary level. A quick reading of this prompt suggests that this event is about understanding an algorithm for solving quadratic equations, but subsequent readings expose the importance of understanding what it means to solve a quadratic equation and understanding a variety of ways to think about solutions and processes for arriving at solutions, which opens the door to a variety of foci.

***Prompt***

In an Algebra 1 class some students began solving a quadratic equation as follows:

 Solve for *x*:

 

They stopped at this point, not knowing what to do next.

*Figure 2*: A prompt from the Solving Quadratic Equations Situation (Situation #35 in the Situations set)

Set of Foci

**Focus**. A focus presents a particular aspect of mathematical understanding for teaching at the secondary level that is relevant to the prompt. Mathematical understanding for teaching includes concepts, processes, representations, solution methods, interpretations, types of reasoning, properties of mathematical objects, and definitions.

**Set of Foci**. A set of foci provides examples of the range and depth of mathematical ideas associated with the prompt.

*Figure 3*: Working descriptions for a focus and set of foci.

The creation of a prompt is followed by the development of its foci, each of which conveys mathematical understanding for teaching as part of the prompt’s set of foci (as noted in Figure 3). The goal of a focus is to unpack the mathematics that may be implicit in the prompt and make explicit the mathematics that could be useful to a teacher who experiences events around content that is similar to that described in the prompt. A group of mathematics educators discussed the prompt, selected mathematics to form into a focus that will promote mathematical understanding, and produced an initial collection of foci. Each focus was designed to communicate with readers who have the equivalent of an undergraduate degree in mathematics or mathematics education.

A prompt can motivate many foci and we selected only a few foci to make a manageable and optimal set for each prompt. Each focus should bring something new and different to the set, but it is reasonable that there may be some overlapping ideas among foci. Selecting a few foci for a Situation is challenging. The foci taken collectively were not meant to be representative of the curriculum, but they were selected to illustrate a broad range of mathematical ideas that connect to the content present in the prompt.

The mathematical topics of the foci for the Solving Quadratic Equations Situation (#35) include such things as meanings of *absolute value*, differences between an unknown and a variable, meanings of *solution*, and the importance and uniqueness of the Zero-Product Property. For example, in Focus 1, we highlight the primacy of equivalent equations in the process of solving equations and connected the ideas to one meaning of absolute value and to the concept of solution of an equation. We point out the Zero-Product Property and its relationship to factoring while considering standard methods of solving quadratic equations by factoring and by employing the quadratic formula in Foci 3 and 4, respectively. In Focus 5, we move to a graphical approach. The set of foci both explore and expand common methods to solve equations as well as attend to an important property and to the meaning of solution and absolute value.

The foci address relevant mathematics but do not address related pedagogical issues. The absence of pedagogical elaborations is not to imply they do not matter but rather to underscore our goal of producing the Situations to develop the MUST framework. It also honors how events similar to the prompt may occur in a variety of classes, though the context of each event will be different. The goal of a focus is not to suggest what a teacher might *do*, but rather to explicate *mathematical understanding* that is useful to a teacher. What the teacher decides to do with the mathematical understanding will vary greatly depending on students and school setting.

Each focus has an introductory, italicized statement that emphasizes the main point of the focus. The italicized statements for the Solving Quadratic Equations Situation (#35) appear in Figure 4. A compete set of the foci for the Situation can be found in chapter XXX. The italicized statements quickly convey the essence of the foci and provide a summary of the set. For example, the italicized statements in Figure 4 suggest that the foci for the Situation, among other things, address several strategies for solving quadratic equations.

***Mathematical Focus 1***

*Factoring and using the Zero Product Property can be used to solve many quadratic equations.*

***Mathematical Focus 2***

*All quadratic equations can be solved by completing the square or by employing the use of the quadratic formula.*

***Mathematical Focus 3***

*A geometric approach that uses area models can be used to represent quadratic equations and their solutions.*

***Mathematical Focus 4***

*Solving an equation through algebraic manipulation requires that equivalence is maintained between each form of the equation.*

***Mathematical Focus 5***

*Equations can be solved by graphically determining the zeros of the associated function.*

*Figure 4*: A list of the Mathematical Foci in Solving Quadratic Equations Situation (#35)

Within a set of foci, a particular focus can take a mathematical direction that complements the mathematics explicit in the prompt, such as by exploring a common algorithm, using a different area of mathematics, or employing a different type of representation. For example, Focus 5 is one of five foci in the Solving Quadratic Equations Situation (#35). It is an example of how a focus can address a prompt with an algebraic or symbolic representation by discussing mathematical understanding of quadratic equations from a graphical point of view. The entire focus appears in Figure 5.

***Mathematical Focus 5***

*Equations can be solved by graphically determining the zeros of the associated function.*

The solutions to an equation in which an expression involving *x* is equal to zero (such as ) are comparable to the zeros of a function of *x* (such as ). This is because a zero, or *x*-intercept, of a function is the *x*-value for which the value of  is zero.

For example, the solutions to  are  and ; and the zeros (the *x*-intercepts) of  are  and . The equation and the function are not the same, as *x* is an unknown in the equation (represents a specific value) while in the function, *x* is a variable (changing quantity). However, the equation and the function are related: the solutions of the equation are the same as the zeros of the function.

Since solutions to equations and zeros of functions are related in this way, graphing a function can be a useful method of solving an equation. However, as noted above, if the strategy is to find the zeros of the function, the accompanying equation must be equal to zero. In this Situation, this will involve manipulating the equation and setting it equal to zero:

 

The graph of *f*(*x*) = *x*2 – *x* – 6 will indicate, by its zeros, the solutions of *x*2 – *x* – 6 = 0.



A similar method requires graphing the functions  and  (that is, treat each side of the original equation as a function) and determine their points of intersection. These are the points at which  and  are equal. We forego this method here, as it is better employed in other Situations.

*Figure 5*: Focus 5 from the Solving Quadratic Equations Situation (#35)

Commentary

**Commentary**. The commentary describes the rationale for or explains the importance of the mathematics in the set of foci.

*Figure 6*: A working description of Commentary.

A Situation can include a commentary (as described in Figure 6) before the set of foci or after the set of foci. The Solving Quadratic Equations Situation (#35) has two commentaries. Regardless of its placement, a commentary provides additional information about the nature of the set of foci that were chosen. For example the (first) commentary shown in Figure 7a provides information about the set of foci and offers a rationale for the selection of foci. As previously explained, the selection of just a few foci is difficult and the commentary offers an opportunity to provide additional information to stimulate the thinking of the reader and to help the reader understand what is important about the selected set as well as what is missing from the set. For example, the post-commentary for the Solving Quadratic Equations Situation (#35) in Figure 7b explains how *domain* is central to equation solving and can be used to explain why extraneous roots arise when various procedures are applied. A commentary also might discuss how the ideas in the foci could apply or generalize to a mathematical context different from that of the prompt. The post commentary in Figure 7b extends extraneous solutions from quadratic and radical settings to logarithmic contexts.

***Commentary***

This Situation provides an opportunity to highlight some issues concerning solving equations (both specifically regarding quadratic equations and in general) that are prevalent in school mathematics.

Foci 1 and 2 present two accurate methods of solving a quadratic equation: factoring and the quadratic formula. These are included because this Prompt illustrates the importance of having accurate and certain means by which to solve quadratic equations. Focus 3 provides a geometric approach for solving . Focus 4 provides guidelines for solving any algebraic equation and emphasizes maintaining equivalence. Focus 5 shows the relationship between the solution(s) of an equation and the zero(s) of a function. This Focus contains a graphical approach to solving quadratic equations. The Situation ends with a Post-Commentary on the occurrence of extraneous solutions.

(a)

***Post Commentary***

*Often, in solving equations, we find extraneous solutions. These result when the original domain is expanded during the course of the solution process.*

It was stated earlier that  and  could not be equal because they have different solution sets, specifically that the former has two solutions and the latter has only one.

To see that  has only one solution, consider how to solve the equation, using the principles in Focus 4.

First, however, attention must be given to the domain of . For  to be defined in the real numbers, . But if , . This means that the domain is the intersection of  and , thus the domain is .

Now, to solve : The inverse operation for taking the square root is squaring. Squaring both sides yields the original problem () and the two solutions are found to be . Note that in this original form of the equation, the domain is the set of real numbers; there are no restrictions on the values of *x*. However, in , , so the only solution is . Negative 2 is called an ***extraneous solution***. It was introduced by expanding the valid domain from the non-negative real numbers to all real numbers.

Note that different problems will have extraneous solutions from the expanding of other domains to the real numbers. Consider . From the first term, . From the second, . The intersection of these two restrictions is . Solving this problem yields two solutions because the problem is converted to a quadratic function with a domain of all real numbers. This domain expansion introduces a possible extraneous solution.

 

In fact, the first solution is not in the original domain and is an extraneous solution. The only solution for this logarithmic equation is .

(b)

*Figure 7*: Example of a commentary and post commentary, from the Solving Quadratic Equations Situation (#35)

Taken collectively, the prompt, foci, and commentary form a Situation that presents an authentic teaching context and explores mathematics that is useful in teaching secondary mathematics. Each Situation connects multiple mathematical ideas to the context, by virtue of how the Situations were created.

Process for Creating Situations

The process of creating Situations began with collecting events and using them to create prompts that stimulated good mathematical discussions. Many of the events were witnessed by supervisors of student teachers as they visited schools and were able to observe classes taught by mentor teachers as well as student teachers. University supervisors were included in hallway discussions and planning sessions. They were often able to capture the verbatim conversation. As we proceeded we realized that there were similar, stimulating events that were remembered from previous interactions with mathematics classes at the secondary level and in university courses preparing teachers to teach mathematics at the secondary level.

A characteristic feature of all of the prompts is that they came from actual events and were not altered to highlight any specific mathematics. Although the prompts were not altered to convey particular mathematics, we chose prompts around which to build complete Situations so that the set would represent the strands of secondary school mathematics. We note that the set might be heavy in some areas, such as algebra and function. Given the large number of algebra courses needed to meet states’ expectations regarding algebra in the secondary school curriculum and the frequency with which student teachers and interns are assigned to algebra classes, it is not surprising that our set of observed events was heavy in examples from algebra and functions.

The next step in creating a Situation was to produce the foci. A team of mathematics educators, including faculty and doctoral students, began to pose and create foci that would illustrate important mathematics. The goal of a focus was to clearly connect secondary school or college mathematics to the prompt in a way that would be informative to a teacher faced with the event, and not to provide a response to the focus. Each focus was reviewed, revised, and discussed in multiple sessions.

Most prompts stimulated the production of a large set of foci, and then the selection of a diverse, small set of foci began. Although we were tempted to include ten to twelve foci for each Situation, we saw the importance of culling the set of foci to approximately four or five critical foci that helped anyone using a Situation to see key components of mathematics that would be part of useful and connected mathematical understanding for teaching secondary school mathematics. With an eye towards taking different mathematics paths, we blended foci with related content and then decided which foci would be most helpful in describing the teacher’s mathematical understanding that would make the most of the learning opportunity. Again, there was extensive debate and often further revision of the set of selected foci.

After the set of foci was determined, we worked to created an italicized topical sentence for each focus to capture the essence of mathematical understanding described in the focus. We used full sentences because our goal was to convey mathematical ideas and not simply to name mathematical topics. A commentary was then written for the set of foci, and occasionally we found that it was necessary to include a second commentary to extend the discussion.

We called upon mathematicians, statisticians, mathematics educators, supervisors of mathematics, and teachers not involved in their creation to vet the set of Situations. Some foci were revised again to increase their clarity or to include a point that was not made explicit.

At all stages of development, we attended carefully to the correctness of the mathematics and the relevance of the mathematics to curricula at the secondary level. The entire set of Situations represents a broad but certainly not complete description of mathematical understanding that is useful to teachers at the secondary level. The set of Situations was sufficiently broad to allow us to develop the MUST framework, which is described in chapter xx.

Attractiveness of the Situations

The primary function of the Situations was to serve as data from actual mathematics instruction that would identify mathematical understanding that is useful at the secondary level. Creating Situations was part of the process of building the MUST Framework. Because the Situations arise from teaching practice and target important mathematics, it was quite natural for those who were creating Situations to use them in their classes and in professional development sessions. As the Situations were used, the foci became better articulated and their use also helped the developers select the critical foci to be included. It was not expected that those who reviewed the Situations would become vested in using the Situations in instruction. However, the Situations were viewed as particularly useful in preparing mathematics teachers at the secondary level and alternative uses soon emerged. A discussion of a variety of uses of Situations is the topic of chapter 4.